Reduction and Exact Solutions of the Ideal Magnetohydrodynamic Equations

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Abstract

In this paper we use the symmetry reduction method to obtain invariant solutions of the ideal magnetohydrodynamic equations in (3+1) dimensions. These equations are invariant under a Galilean-similitude Lie algebra for which the classification by conjugacy classes of r-dimensional subalgebras ($1 \leq r \leq 4$) was already known. So we restrict our study to the three-dimensional Galilean-similitude subalgebras that give systems composed of ordinary differential equations. We present here several examples of these solutions. Some of these exact solutions show interesting physical interpretations.

PACS numbers: 02.20.Qs, 02.30.Jr, 47.65.+a
Mathematics Subject Classification: 76M60, 35C05

1 Introduction

The ideal magnetohydrodynamic (MHD) equations are the most basic single-fluid model with the Maxwell's equations for describing the interactions between magnetic and pressure forces within an electrically conducting fluid for long spatial scale and low frequency phenomena in plasma physics. There are not many exact solutions in the theory of ideal magnetized fluid governed by the system of MHD equations. We present here some exact analytic solutions of this system. By exact solution we mean one obtained by a reduction of the full set of partial differential equations (PDEs) of the original system to one or more ordinary differential equations (ODEs) involving fewer independent variables. This can be achieved by applying the symmetry reduction method (SRM) based on group theory. This method for reducing the number of independent variables in an equation is to require that a solution be invariant under some subgroups of the Lie symmetry group of the given system; the so-called G-invariant solution (GIS).

The paper is organized as follows. Section 2 briefly gives a general description of the ideal MHD equations. In Section 3 contains a short summary of the SRM and its application to the MHD system. In Sections 4 and 5, some examples of GIS are presented and also interpreted by a physical point of view for two configurations of the magnetic field: $\mathbf{B} = (\mathrm{B}_1, \mathrm{B}_2, 0)$ and $\mathbf{B} = (\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3)$, respectively. Section 6 summarizes the obtained results which suggest further applications.

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2 Preliminary information

The fluid under consideration is assumed to be ideal and perfectly conductive, that is, neglecting all the dissipative and dispersive effects, such as viscosity, magnetic resistivity, thermal conductivity and Hall effect. The fluid is assumed to be unbounded and the medium is non magnetic, so its magnetic permeability μ is taken to be one. Furthermore we suppose that the fluid is described by a perfect gas equation of state. Under the above assumptions the ideal MHD model is governed by the system of nine PDEs:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + (\nabla\cdot\mathbf{v})\rho = 0\,, \tag{2.1}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \rho^{-1}\nabla p + \rho^{-1}(\mathbf{B}\times\mathbf{J}) = \mathbf{0}\,, \tag{2.2}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \gamma(\nabla\cdot\mathbf{v})p = 0\,, \tag{2.3}$$

$$\frac{\partial\mathbf{B}}{\partial t} - \nabla\times\big(\mathbf{v}\times\mathbf{B}\big) = \mathbf{0}\,, \tag{2.4}$$

$$\nabla\cdot\mathbf{B} = 0\,, \tag{2.5}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + (\mathbf{v}\cdot\nabla)$ is the convective derivative. Here ρ is the mass density, p is the pressure, $\mathbf{v} = \big(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\big)$ and $\mathbf{B} = \big(\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3\big)$ are the flow velocity and the magnetic field, respectively and γ is the adiabatic exponent. Note that throughout this paper we denoted by $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ the unit vectors in the direction of the x-, y- and z-axes, respectively. The current density is given by the Ampère's law

$$\mathbf{J} = \nabla\times\mathbf{B}\,, \tag{2.6}$$

for which the displacement current is negligible since the flow is non-relativistic. All the functions $\mathrm{u} = \big(\rho, p, \mathbf{v}, \mathbf{B}\big)$ depend on time t and coordinates $\mathbf{x} = (x, y, z)$. The system of MHD equations (2.1)–(2.5) is quasilinear, hyperbolic, and is written in the normal (Cauchy-Kowalewski) form. In equation (2.2), there is no extraneous force other than the Lorentz force (i.e., the magnetic force)

$$\mathbf{F_m} = \mathbf{J}\times\mathbf{B}\,, \tag{2.7}$$

which can be broken down into a magnetic pressure force and a magnetic tension force

$$\mathbf{F_m} = -\frac{1}{2}\nabla\big(|\mathbf{B}|^2\big) + (\mathbf{B}\cdot\nabla)\mathbf{B}\,. \tag{2.8}$$

It is noteworthy to recall that the integral form of Faraday's law (2.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{S}}\mathbf{B}\cdot\mathrm{d}\mathbf{s}\right) = 0\,, \tag{2.9}$$

where $\mathcal{S}$ is an arbitrary surface moving with the fluid, corresponds to the magnetic flux conservation law, i.e., the "Alfvén's frozen-in" theorem [1], and has some fundamental consequences. The magnetic field lines are "glued" to the medium, and transported entirely by convection and there is no diffusion of $\mathbf{B}$ through the conducting media. So, in "flux-freezing" regime, the convective derivative of $\mathbf{B}/\rho$ can be written as [2]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) - \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)\mathbf{v} = \mathbf{0}\,. \tag{2.10}$$

This says that $\mathbf{B}/\rho$ evolves in the same manner as the separation $\boldsymbol{\delta l}$ between two points in the fluid. Notice that the vorticity $\boldsymbol{\omega}$ of the flow and $\mathbf{B}$ are physically related by:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla\times\left(\mathbf{v}\times\boldsymbol{\omega}\right) = \nabla\times\left[\frac{\mathbf{F_m}}{\rho}\right]. \tag{2.11}$$

In hydrodynamics where $\mathbf{B}=\mathbf{0}$, if at one instance $\boldsymbol{\omega}=\mathbf{0}$, the flow will be irrotational for all subsequent times. So, the vorticity field obeys a transport equation similar to (2.4): the vortex lines are frozen into the fluid matter. However this result does not hold for ideal MHD because the term $\nabla\times[\mathbf{F_m}/\rho]$ is in general not equal to zero, and may induce vorticity even though the flow is initially irrotational.

3 The symmetry of the magnetohydrodynamic equations

The language of group theoretical methods for studying systems of PDEs is a very useful and suitable tool for investigating the main features of numerous problems in various branches of mathematical physics. The task of finding an increasing number of solutions of systems of PDEs is related to the group properties of these differential equations. Its main advantages appear when group analysis makes it possible to construct regular algorithms for finding certain classes of solutions without referring to additional considerations but proceeding only from the given systems of PDEs. A systematic computational method for constructing the group of symmetries for given system of PDEs has been extensively developed by many authors [3]–[7]. A broad review of recent developments in this subject can be found in several books as [6], [7]. The methodological approach adopted here is based on the symmetry reduction method for PDEs invariant under a Lie group $\mathcal{G}$ of point transformations. This means that the groups under considerations are the connected local transformation groups acting on the space of p independent and q dependent variables $E\times U$, i.e.,

$$\widetilde{\mathrm{x}} = \Lambda_{\mathcal{G}}(\mathrm{x},\mathrm{u}), \qquad \widetilde{\mathrm{u}} = \Omega_{\mathcal{G}}(\mathrm{x},\mathrm{u})\,; \qquad \mathrm{x}=(x^1,\dots,x^p), \qquad \mathrm{u}=(u^1,\dots,u^q). \tag{3.1}$$

By a symmetry group of a system Δ of differential equations we understand a local Lie group $\mathcal{G}$ transforming both the independent and dependent variables of the considered system of equations in such a way that $\mathcal{G}$ transforms given solutions $\mathrm{u}(\mathrm{x})$ of Δ to the new ones $\widetilde{\mathrm{u}}(\widetilde{\mathrm{x}})$ of Δ. The Lie algebra $\mathcal{L}$ of $\mathcal{G}$ is realized by the vector fields $\hat{X}$ which for the system composed of m PDEs of order k

$$\Delta^l\left(\mathrm{x},\mathrm{u}^{(k)}\right) = 0, \qquad l=1,\dots,m \tag{3.2}$$

where $\mathrm{u}^{(k)}$ denotes all partial derivatives of u up to order k, can be expressed as follows

$$\hat{X} = \xi^\mu(\mathrm{x},\mathrm{u})\partial_{x^\mu} + \phi^j(\mathrm{x},\mathrm{u})\partial_{u^j}\,, \tag{3.3}$$

where ξ^μ and ϕ^j are assumed to be functions of $(\mathrm{x},\mathrm{u})\in E\times U$ only. The functions ξ^μ and ϕ^j are defined by the infinitesimal invariance criterion [5]

$$\mathrm{pr}^{(k)}(\hat{X})\Delta^l\Big|_{\Delta^n=0} = 0, \qquad l,n=1,\dots,m \tag{3.4}$$

where $\mathrm{pr}^{(k)}(\hat{X})$ is the k prolongation of the vector field. There exist standard algorithms for determining the symmetry algebra $\mathcal{L}$ and classifying subalgebras $\mathcal{L}_p$ of $\mathcal{L}$ [7].

The symmetry algebra of the MHD equations (2.1)–(2.5), denoted by $\mathcal{M}$, has been found by Fuchs [8], and independently by Grundland and Lalague [9]. It is spanned by the following 13 Galilean-similitude (GS) infinitesimal generators:

$$\begin{aligned} &P_\mu = \partial_{x_\mu}\,, \qquad J_k = \epsilon_{kij}\big(x_i\partial_{x_j} + \mathrm{v}_j\partial_{\mathrm{v}_j} + \mathrm{B}_i\partial_{\mathrm{B}_j}\big)\,, \qquad K_i = t\partial_{x_i} + \partial_{\mathrm{v}_i}\,, \\ &F = t\partial_t + x_i\partial_{x_i}\,, \quad G = -t\partial_t - 2\rho\partial_\rho + \mathrm{v}_i\partial_{\mathrm{v}_i}\,, \quad H = 2\rho\partial_\rho + 2p\partial_p + \mathrm{B}_i\partial_{\mathrm{B}_i}\,, \end{aligned} \tag{3.5}$$

where ϵ_{kij} is the Levi-Civita symbol, $i, j, k = 1, 2, 3$; $\mu = 0, 1, 2, 3$. Thus, the MHD equations (2.1)–(2.5) are invariant under time (P_o) and spatial (P_i) translations, rotations (J_k), Galilei transformations (K_i), dilations (F, G, and H which is the center of $\mathcal{M}$; H commutes with all the generators). The basis of $\mathcal{M}$ and the commutator tables are given in [9]. In contrast to the results obtained for the (1+1) and (2+1) dimensional versions of the MHD model [8], the dimension of the Lie algebra $\mathcal{M}$ for MHD equations (2.1)–(2.5) in the full (3+1) dimensions (x, y, z and time t) is independent of the value of the adiabatic exponent γ that we treat here as a fixed parameter: $\gamma \geq 1$.

A classification by conjugacy classes of r-dimensional subalgebras ($1 \leq r \leq 4$) corresponding to the GS algebra $\mathcal{M}$ has been established by Grundland and Lalague [9]. Their investigation is limited to subgroups of dimension not greater than the number of independent variables. This constraint is imposed by the SRM itself, in order to reduce the initial system of PDEs to a lower dimensional system of PDEs which now involves $p - r$ independent variables. The method of classification which they have used was developed by Patera *et al* [10], and published in a series of papers in the late 1970's. More recently, this subject has been treated in such books as [6], [7].

The aim of this work [11] was to fulfill the last steps of the SRM applied to the MHD equations (2.1)–(2.5). Obtaining the invariant solutions, called G-invariant solutions (GIS), under some subgroup $\mathcal{G}_p \subset \mathcal{G}$ requires the knowledge of the group invariant $\mathcal{I}$ of $\mathcal{G}_p$. For each of the 104 three-dimensional subalgebras of $\mathcal{M}$ taken from [9], we have calculated a set of functionally independent invariants. Suppose $\{v_1, v_2, v_3\}$ is a basis of infinitesimal generators of the Lie algebra $\mathcal{L}_3$, then $\mathcal{I}$ is a $\mathcal{L}_3$-invariant if and only if $v_i(\mathcal{I}) = 0$, $i = 1, 2, 3$. Thus, for each algebra, we obtain nine functionally invariants. We can calculate invariant solutions if the invariants are of the form

$$\mathcal{I} = \big\{s(\mathrm{x})\,;\, I_i\,(\mathrm{x}, \mathrm{u})\big\}, \qquad i = 1, \ldots, 8 \tag{3.6}$$

such that

$$\mathrm{rank}\left(\frac{\partial I_i\,(\mathrm{x}, \mathrm{u})}{\partial \mathrm{u}}\right) = 8\,. \tag{3.7}$$

In our investigation, we find that the invariants of twelve three-dimensional algebras among those classified by Grundland and Lalague [9] do not have the last properties. The solutions corresponding to these algebras are called partially invariant solutions (PIS) and therefore the SRM is inapplicable. So, we concentrate here only on invariant solutions.

From the expression given in (3.6), we form the equation

$$I_i\,(\mathrm{x}, \mathrm{u}) = \mathcal{F}_i\,\big(s(\mathrm{x})\big)\,, \qquad i = 1, \ldots, 8 \tag{3.8}$$

where $\mathcal{F}_i$ are arbitrary functions, giving the orbits of an invariant solution. From (3.7) and the implicit functions theorem, we obtain

$$\mathrm{u} = \phi_i\big(\mathcal{F}_j\,(s), \mathrm{x}\big)\,, \qquad i, j = 1, \ldots, 8 \tag{3.9}$$

where ϕ_i are arbitrary functions. Substituting each of the expressions (3.9) into equations (2.1)–(2.5), we can reduce the MHD system to a new system composed of ODEs. Such system is called the reduced system. The variable s is the symmetry variable and plays the role of the independent variable of the reduced system. Solving the reduced system and inserting these solutions into (3.9), we obtain finally the GIS of the system (2.1)–(2.5).

In the two next sections, we present ten examples of GIS that we have obtained from the GS subalgebras of dimension three [9] (G_i refers to the solution corresponding to this algebra given in the title of each subsection). These solutions are also classified by some of their physical characteristics and the types of the waves that they may generate.

4 $\mathbf{B} = \big(\mathbf{B_1}, \mathbf{B_2}, \mathbf{0}\big)$

$\boldsymbol{G_1 = \big\{J_3 + K_3 + \alpha H,\ P_1,\ P_2\big\}}, \qquad \alpha \in \mathbb{R}$

From this algebra, we have obtained the corresponding G-invariant solution

$$\begin{aligned}
&\rho = \exp\left[2\alpha\frac{z}{t}\right] R(t), \qquad p = A_o\, t^{(1-\gamma)}\rho\,, \qquad\qquad (4.1)\\
&\mathrm{v}_1 = U_o \sin\left(\frac{1}{2\alpha}\ln\left[\frac{\theta_o}{tR(t)}\right] - \frac{z}{t}\right), \ \ \mathrm{v}_2 = U_o \cos\left(\frac{1}{2\alpha}\ln\left[\frac{\theta_o}{tR(t)}\right] - \frac{z}{t}\right), \ \ \mathrm{v}_3 = \frac{z}{t} - \frac{W(t)}{t}\,,\\
&\mathrm{B}_1 = X_o\sqrt{\frac{\rho}{t}}\sin\left(\frac{1}{2\alpha}\ln\left[\frac{\theta_o}{tR(t)}\right] - \frac{z}{t}\right), \ \ \mathrm{B}_2 = X_o\sqrt{\frac{\rho}{t}}\cos\left(\frac{1}{2\alpha}\ln\left[\frac{\theta_o}{tR(t)}\right] - \frac{z}{t}\right), \ \ \mathrm{B}_3 = 0\,,
\end{aligned}$$

where U_o, X_o and $\theta_o \in \mathbb{R}$. Note that throughout this paper, we denoted the arbitrary constants by $(\ )_o$. The functions R and W are given by:

$$R(t) = \begin{cases} R_o\, t^{(4\alpha^2 A_o - 1)} \exp\left[-\dfrac{2\alpha}{t}W_o - \dfrac{2\alpha^2}{t}X_o^2\big(1+\ln[t]\big)\right] & \text{for } \ \gamma = 1\,,\\ \dfrac{R_o}{t}\exp\left[-\dfrac{2\alpha}{t}W_o - \dfrac{2\alpha^2}{t}\big(2A_o + X_o^2\big)\big(1+\ln[t]\big)\right] & \text{for } \ \gamma = 2\,,\\ \dfrac{R_o}{t}\exp\left[-\dfrac{2\alpha}{t}W_o - \dfrac{2\alpha^2}{t}X_o^2\big(1+\ln[t]\big) + \dfrac{4\alpha^2 A_o}{(2-\gamma)(1-\gamma)}t^{(1-\gamma)}\right] & \text{for } \gamma \neq 1, 2\,, \end{cases}$$

$$W(t) = \begin{cases} W_o + \alpha\big(2A_o + X_o^2\big)\ln[t] \ \ \text{for} \ \ \gamma = 2\,,\\ W_o + \dfrac{2\alpha A_o}{(2-\gamma)}t^{(2-\gamma)} + \alpha X_o^2 \ln[t] \ \ \text{for} \ \ \gamma \neq 2\,, \end{cases}$$

with $R_o \in \mathbb{R}^+$, $\alpha \in \mathbb{R}/\{0\}$; $0 \leq A_o < 1/4\alpha^2$ for $\gamma = 1$ and $A_o \in \mathbb{R}^+$ for $\gamma > 1$, $W_o \in \mathbb{R}$ with $\mathrm{sgn}[\mathrm{W_o}] = \mathrm{sgn}[\alpha]$. Under these conditions and for $t > 0$, the solution G_1 is non-singular and tends to zero for large values of t.

This solution can be interpreted physically as a nonstationary and compressible fluid for which the shape of the flow is a helix of radius U_o and pitch angle $\phi = \arctan\big[\mathrm{v}_3/U_o\big]$. Indeed, the motion described by the solution G_1 is a circular motion in the x-y plane (where lies the magnetic field $\mathbf{B}$) and an accelerated motion parallel to the z-axis resulting from the action of the Lorentz force:

$$\mathbf{F_m} = -\frac{1}{2}\nabla\big(|\mathbf{B}|^2\big) = -\alpha X_o^2\frac{\rho}{t^2}\,\mathbf{e}_3\,. \qquad (4.2)$$

The Lorentz force $\mathbf{F_m}$ acts perpendicularly to $\mathbf{B}$, causing compressions and expansions of the distance between lines of force without changing their direction, as does a magnetoacoustic fast wave (F) which propagates perpendicularly to $\mathbf{B}$ in an ideal fluid [12]. There is no tension force acting on the magnetic fields lines. So, the classical Bernoulli theorem is still valid with the pressure of the fluid is replaced by $p + (|\mathbf{B}|^2/2)$ which is then the total pressure. The solution G_1 has vorticity that lies in the x-y plane, and referring to (2.11), the vortex lines move with the fluid. Consequently, by virtue of Kelvin's theorem the circulation

$$\Gamma_c = \oint_C \mathbf{v} \cdot \mathrm{d}\boldsymbol{l} \tag{4.3}$$

around a fluid element is preserved [13]. Also, the solution G_1 is characterized by the fact that $(\mathbf{B} \cdot \nabla)\mathbf{v} = \mathbf{0}$, which shows that the velocity of the fluid $\mathbf{v}$ is constant along each line of force but also meaning that $\mathbf{B}/\rho$ is conserved along the flow, and consequently the magnetic field lines remains inextensible [c.f., (2.10)]. The current density induced by the magnetic field $\mathbf{B}$ is equal to

$$\mathbf{J} = -\frac{1}{t}\big[\alpha \mathrm{B}_2 + \mathrm{B}_1\big]\mathbf{e}_1 + \frac{1}{t}\big[\alpha \mathrm{B}_1 - \mathrm{B}_2\big]\mathbf{e}_2\,, \tag{4.4}$$

which lies in the x-y plane and always has a nonzero component.

$\boldsymbol{G_2 = \{J_3 + K_3 + \alpha_1 P_3 + \alpha_2 H,\ K_1,\ K_2\}}, \quad \alpha_1,\ \alpha_2 \in \mathbb{R}$

With respect to this algebra, we have calculated the corresponding invariant solution

$$\begin{aligned}
\rho &= \frac{R_o}{t^2(t+\alpha_1)} \exp\left[R(t) - \frac{[5\alpha_1 + W_o - 2\alpha_2 z]}{(t+\alpha_1)} - \frac{2\alpha_2^2 X_o^2}{(t+\alpha_1)}\big(1 + \ln[t+\alpha_1]\big)\right],\\
p &= A_o\big[t^2(t+\alpha_1)\big]^{(1-\gamma)}\rho\,,\\
\mathrm{v}_1 &= \frac{x}{t} - \frac{U_o}{t}\sin\theta(z,t), \qquad \mathrm{v}_2 = \frac{y}{t} - \frac{U_o}{t}\cos\theta(z,t)\,,\\
\mathrm{v}_3 &= \frac{z}{(t+\alpha_1)} - \frac{2\alpha_2}{(t+\alpha_1)}W(t) - \frac{A_o\alpha_2 X_o^2}{(t+\alpha_1)}\ln[t+\alpha_1] - \frac{(5\alpha_1 + W_o)}{2\alpha_2(t+\alpha_1)}\,,\\
\mathrm{B}_1 &= \frac{X_o\sqrt{\rho}}{\sqrt{t+\alpha_1}}\sin\theta(z,t), \qquad \mathrm{B}_2 = \frac{X_o\sqrt{\rho}}{\sqrt{t+\alpha_1}}\cos\theta(z,t), \qquad \mathrm{B}_3 = 0\,,
\end{aligned} \tag{4.5}$$

where R_o, $A_o \in \mathbb{R}^+$; W_o, U_o and $X_o \in \mathbb{R}$. The expressions for the functions R and W depend on specific values of γ.

For $\gamma = 3/2$:

$$\begin{aligned}
R(t) &= \left(\frac{\sqrt{t+\alpha_1} - \sqrt{\alpha_1}}{\sqrt{t+\alpha_1} + \sqrt{\alpha_1}}\right)^{2\alpha_2^2 A_o/(\alpha_1)^{3/2}} \exp\left[\frac{4\alpha_2^2 A_o}{\alpha_1\sqrt{(t+\alpha_1)}} + \frac{4\alpha_2^2 A_o}{\sqrt{\alpha_1}}\frac{\operatorname{arc\,tanh}\left[\sqrt{1+\frac{t}{\alpha_1}}\right]}{(t+\alpha_1)}\right],\\
W(t) &= -\frac{2}{\sqrt{\alpha_1}}\operatorname{arc\,tanh}\left[\sqrt{1+\frac{t}{\alpha_1}}\right].
\end{aligned}$$

For $\gamma = 2$:

$$R(t) = \left(1+\frac{\alpha_1}{t}\right)^{8\alpha_2^2 A_o/\alpha_1^3} \exp\left[-\frac{4\alpha_2^2 A_o}{\alpha_1^2}\left(\frac{2+\ln\left[1+\frac{\alpha_1}{t}\right]}{(t+\alpha_1)}\right)\right],$$
$$W(t) = \frac{1}{\alpha_1^2}\ln\left[1+\frac{\alpha_1}{t}\right] - \frac{1}{\alpha_1 t}.$$

And otherwise for $\gamma \neq 3/2,\, 2$:

$$W(t) = \frac{\alpha_1^{(1-\gamma)} t^{(3-2\gamma)}}{(3-2\gamma)}\,{}_2F_1\left(3-2\gamma, \gamma, 4-2\gamma, \frac{-t}{\alpha_1}\right) + \frac{\alpha_1^{-\gamma} t^{(4-2\gamma)}}{(4-2\gamma)}\,{}_2F_1\left(4-2\gamma, \gamma, 5-2\gamma, \frac{-t}{\alpha_1}\right),$$
$$R(t) = \int^t \frac{W(s)}{(s+\alpha_1)^2}\mathrm{d}s\,,$$

where ${}_2F_1$ denotes the hypergeometric function of the second kind. The expression for the function θ is given by

$$\theta(t) = \theta_o - \frac{R(t)}{2\alpha_2} + \frac{[5\alpha_1 + W_o - 2\alpha_2 z]}{2\alpha_2(t+\alpha_1)} + \frac{\alpha_2 X_o^2}{(t+\alpha_1)}\big(1+\ln[t+\alpha_1]\big)\,, \qquad \theta_o \in \mathbb{R}\,. \tag{4.6}$$

Solution G_2 is always real and non-singular if $t > 0$, $\alpha_1 > 0$ and $\alpha_2 \neq 0$, and tends asymptotically to zero when $t \to \infty$. It describes a nonstationary and compressible flow in (3+1) dimensions with vorticity that lies in the x-y plane. The current density induced by (4.5) is equal to

$$\mathbf{J} = -\frac{[\alpha_2 \mathrm{B}_2 + \mathrm{B}_1]}{(t+\alpha_1)}\,\mathbf{e}_1 + \frac{[\alpha_2 \mathrm{B}_1 - \mathrm{B}_2]}{(t+\alpha_1)}\mathbf{e}_2\,, \tag{4.7}$$

and the resulting Lorentz force takes the form

$$\mathbf{F_m} = -\frac{\alpha_2 X_o^2}{(t+\alpha_1)^2}\,\rho\,\mathbf{e}_3\,. \tag{4.8}$$

Since $\mathbf{F_m}$ is a conservative force (i.e., it can be derived from the gradient of the magnetic pressure $|\mathbf{B}|^2/2$) implies, by virtue of Kelvin's theorem, that the circulation of the fluid is conserved.

$\boldsymbol{G_3 = \{J_3 + P_3 + \alpha_1 G + \alpha_2 H,\ K_1,\ K_2\}}$, $\quad \alpha_1 \neq 0$, $\alpha_2 \in \mathbb{R}$

From this algebra, we have computed the corresponding analytical solution

$$\begin{aligned}
&\rho = \frac{R_o}{W} t^{2(\alpha_1-\alpha_2)/\alpha_1} \exp\left[2\left(\frac{\alpha_2}{\alpha_1}-1\right)\mathcal{F}(s)\right], \quad p = \frac{A_o}{W^\gamma} t^{-2\alpha_2/\alpha_1} \exp\left[2\left(\frac{\alpha_2}{\alpha_1}-\gamma\right)\mathcal{F}(s)\right],\\
&\mathrm{v}_1 = \frac{x}{t} - \frac{U_o}{t}\sin\big[\theta(s)\big], \qquad \mathrm{v}_2 = \frac{y}{t} - \frac{U_o}{t}\cos\big[\theta(s)\big], \qquad \mathrm{v}_3 = \frac{1}{t}\big[W - 1/\alpha_1\big]\,,\\
&\mathrm{B}_1 = \frac{X_o}{W} t^{-\alpha_2/\alpha_1} \exp\left[\left(\frac{\alpha_2}{\alpha_1}-\frac{1}{\alpha_1}\right)\mathcal{F}(s)\right]\sin\big[\theta(s)\big]\,,\\
&\mathrm{B}_2 = \frac{X_o}{W} t^{-\alpha_2/\alpha_1} \exp\left[\left(\frac{\alpha_2}{\alpha_1}-\frac{1}{\alpha_1}\right)\mathcal{F}(s)\right]\cos\big[\theta(s)\big], \qquad \mathrm{B}_3 = 0\,,
\end{aligned} \tag{4.9}$$

where R_o, A_o, U_o, V_o and X_o are arbitrary constants. The functions $\mathcal{F}$ and θ are given by

$$\mathcal{F} = \int^s \frac{\mathrm{d}s'}{W}\,, \qquad \theta = \theta_o + \frac{\ln t}{\alpha_1} - \frac{\mathcal{F}}{\alpha_1}\,, \qquad \theta_o \in \mathbb{R}\,, \tag{4.10}$$

which both depend on the symmetry variable $s = z + (\ln t)/\alpha_1$. The unknown function W is determined by solving the reduced form of the equation (2.2) along the z-axis:

$$\begin{aligned} W^3\frac{\mathrm{d}W}{\mathrm{d}s} - W^3 + \frac{W^2}{\alpha_1} - \frac{A_o}{R_o}\left[\gamma\frac{\mathrm{d}W}{\mathrm{d}s} + 2\left(\gamma - \frac{\alpha_2}{\alpha_1}\right)\right] W^{(2-\gamma)} \exp\left[2(2-\gamma)\mathcal{F}(s)\right] \\ - \frac{X_o^2}{R_o}\left[\frac{\mathrm{d}W}{\mathrm{d}s} + \left(\frac{1}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\right)\right] \exp\left[\left(4 - \frac{2}{\alpha_1}\right)\mathcal{F}(s)\right] = 0\,. \end{aligned} \tag{4.11}$$

Equation (4.11) is difficult to solve completely. However, with the ansatz,

$$W = C_1 s + C_2, \qquad C_1,\ C_2 \in \mathbb{R}, \qquad C_1 \neq 0\,, \tag{4.12}$$

we can transform (4.11) into a polynomial equation

$$(C_1 - 1)W^3 + \frac{W^2}{\alpha_1} - \frac{A_o}{R_o}\left[(C_1+2)\gamma - \frac{2\alpha_2}{\alpha_1}\right] W^m - \frac{X_o^2}{R_o}\left[C_1 + \left(\frac{1}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\right)\right] W^n = 0\,.$$

where $m = (2-\gamma)(C_1+2)/C_1$ and $n = 2(2\alpha_1 - 1)/(\alpha_1 C_1)$. By equating the coefficients of the same power of W, we obtain five cases of invariant solutions of Eqs (2.1)–(2.5).

Case 1.) $C_1 = 1, \qquad \gamma = 4/3, \qquad \alpha_2 = \alpha_1 + 1$

$$\begin{aligned} &\rho = R_o t^{-2/\alpha_1}\, W_1^{(2-3\alpha_1)/\alpha_1}, \quad p = \frac{R_o}{2(\alpha_1 - 1)} t^{-2(\alpha_1+1)/\alpha_1}\, W_1^{2(1-\alpha_1)/\alpha_1}, \\ &\mathrm{v}_1 = \frac{x}{t} - \frac{U_o}{t}\sin\theta_1, \qquad \mathrm{v}_2 = \frac{y}{t} - \frac{U_o}{t}\cos\theta_1, \qquad \mathrm{v}_3 = \frac{1}{t}\left[W_1 - \frac{1}{\alpha_1}\right], \\ &\mathrm{B}_1 = X_o\, t^{-(\alpha_1+1)/\alpha_1}\sin\theta_1, \qquad \mathrm{B}_2 = X_o\, t^{-(\alpha_1+1)/\alpha_1}\cos\theta_1, \qquad \mathrm{B}_3 = 0\,, \end{aligned} \tag{4.13}$$

where $R_o \in \mathbb{R}^+$, U_o, $X_o \in \mathbb{R}$; $\alpha_1 > 0$. The functions W_1 and θ_1 are given by

$$W_1 = \frac{\ln[t]}{\alpha_1} + z + C_2, \qquad \theta_1 = \theta_o - \frac{1}{\alpha_1}\ln\left[\frac{W_1}{t}\right], \qquad C_2,\ \theta_o \in \mathbb{R}\,.$$

Case 2.) $C_1 = 1, \qquad \gamma = 2\alpha_2/3, \qquad \alpha_1 = 1$

$$\begin{aligned} &\rho = R_o\, t^{2(1-\alpha_2)}\, W_2^{(2\alpha_2-5)}, \qquad p = A_o\, t^{-2\alpha_2}\,, \\ &\mathrm{v}_1 = \frac{x}{t} - \frac{U_o}{t}\sin\theta_2, \qquad \mathrm{v}_2 = \frac{y}{t} - \frac{U_o}{t}\cos\theta_2, \qquad \mathrm{v}_3 = \frac{1}{t}\left[W_2 - 1\right], \\ &\mathrm{B}_1 = \frac{\sqrt{R_o}}{\sqrt{(2-\alpha_2)}}\, t^{-\alpha_2}\, W_2^{(\alpha_2-2)}\sin\theta_2\,, \\ &\mathrm{B}_2 = \frac{\sqrt{R_o}}{\sqrt{(2-\alpha_2)}}\, t^{-\alpha_2} W_2^{(\alpha_2-2)}\cos\theta_2, \qquad \mathrm{B}_3 = 0\,, \end{aligned} \tag{4.14}$$

where $R_o,\ A_o \in \mathbb{R}^+$, $U_o,\ X_o \in \mathbb{R}$; $3/2 \le \alpha_2 < 2$. The functions W_2 and θ_2 take the form

$$W_2 = \ln[t] + z + C_2, \qquad \theta_2 = \theta_o - \ln\left[\frac{W_2}{t}\right] \qquad C_2,\ \theta_o \in \mathbb{R}\,.$$

Case 3.) $C_1 = 1, \qquad \gamma = 4/3, \qquad \alpha_1 = 1$

$$\begin{aligned}
&\rho = (2-\alpha_2)\left[2A_o + X_o^2\right] t^{2(1-\alpha_2)}\, W_2^{(2\alpha_2-5)}, \qquad p = A_o t^{-2\alpha_2}\, W_2^{(2\alpha_2-4)}\,, \\
&\mathrm{v}_1 = \frac{x}{t} - \frac{U_o}{t}\sin\theta_2, \qquad \mathrm{v}_2 = \frac{y}{t} - \frac{U_o}{t}\cos\theta_2, \qquad \mathrm{v}_3 = \frac{1}{t}\left[W_2 - 1\right], \\
&\mathrm{B}_1 = X_o t^{-\alpha_2}\, W_2^{(\alpha_2-2)}\sin\theta_2, \qquad \mathrm{B}_2 = X_o t^{-\alpha_2}\, W_2^{(\alpha_2-2)}\cos\theta_2, \qquad \mathrm{B}_3 = 0\,,
\end{aligned} \tag{4.15}$$

where $A_o \in \mathbb{R}^+$, $U_o,\ X_o \in \mathbb{R}$; $1 < \alpha_2 < 2$. The functions W_2 and θ_2 retain the form given in solution (4.14).

Case 4.) $C_1 = (2\alpha_1 - 1)/\alpha_1\,, \qquad \gamma = (2\alpha_1+1)/(4\alpha_1-1)$

$$\begin{aligned}
\rho &= R_o\, t^{2(\alpha_1-\alpha_2)/\alpha_1}\, W_3^{(2\alpha_2-6\alpha_1+1)/(2\alpha_1-1)}\,, \\
p &= \frac{(1-\alpha_1)R_o}{[2(\alpha_2-\alpha_1)-1]}\, t^{-2\alpha_2/\alpha_1} W_3^{(2\alpha_2-2\alpha_1-1)/(2\alpha_1-1)}\,, \\
\mathrm{v}_1 &= \frac{x}{t} - \frac{U_o}{t}\sin\theta_3, \qquad \mathrm{v}_2 = \frac{y}{t} - \frac{U_o}{t}\cos\theta_3, \qquad \mathrm{v}_3 = \frac{1}{t}\left[W_3 - \frac{1}{\alpha_1}\right], \\
\mathrm{B}_1 &= \frac{\sqrt{R_o}}{\sqrt{(2\alpha_1-\alpha_2)}}\, t^{-\alpha_2/\alpha_1} W_3^{(\alpha_2-2\alpha_1)/(2\alpha_1-1)}\sin\theta_3\,, \\
\mathrm{B}_2 &= \frac{\sqrt{R_o}}{\sqrt{(2\alpha_1-\alpha_2)}}\, t^{-\alpha_2/\alpha_1} W_3^{(\alpha_2-2\alpha_1)/(2\alpha_1-1)}\cos\theta_3, \qquad \mathrm{B}_3 = 0\,,
\end{aligned} \tag{4.16}$$

where $R_o \in \mathbb{R}^+$, $U_o \in \mathbb{R}$. The parameters α_1 and α_2 lie in these intervals: $1/2 < \alpha_1 < 1$, $2\alpha_1 < \alpha_2 < (2\alpha_1+1)/2$. The functions W_3 and θ_3 take the form

$$W_3 = \frac{(2\alpha_1-1)}{\alpha_1}\left[\frac{\ln[t]}{\alpha_1} + z\right] + C_2, \qquad \theta_3 = \theta_o + \frac{\ln[t]}{\alpha_1} + \frac{\ln[W_3]}{(1-2\alpha_1)}, \quad C_2,\ \theta_o \in \mathbb{R}\,.$$

Case 5.) $C_1 = (4\alpha_1 - 2)/3\alpha_1\,, \qquad \gamma = 6\alpha_1/(5\alpha_1 - 1)$

$$\begin{aligned}
\rho &= R_o\, t^{2(\alpha_1-\alpha_2)/\alpha_1} W_4^{(3\alpha_2-8\alpha_1+1)/(2\alpha_1-1)}\,, \\
p &= \frac{R_o}{2(2\alpha_1-\alpha_2)} t^{-2\alpha_2/\alpha_1} W_4^{3(\alpha_2-2\alpha_1)/(2\alpha_1-1)}\,, \\
\mathrm{v}_1 &= \frac{x}{t} - \frac{U_o}{t}\sin\theta_4, \quad \mathrm{v}_2 = \frac{y}{t} - \frac{U_o}{t}\cos\theta_4, \quad \mathrm{v}_3 = \frac{1}{t}\left[W_4 - \frac{1}{\alpha_1}\right], \\
\mathrm{B}_1 &= \sqrt{\frac{(\alpha_1-2)R_o}{(4\alpha_1-3\alpha_2+1)}}\, t^{-\alpha_2/\alpha_1} W_4^{(3\alpha_2-4\alpha_1-1)/(4\alpha_1-2)}\sin\theta_4\,, \\
\mathrm{B}_2 &= \sqrt{\frac{(\alpha_1-2)R_o}{(4\alpha_1-3\alpha_2+1)}}\, t^{-\alpha_2/\alpha_1} W_4^{(3\alpha_2-4\alpha_1-1)/(4\alpha_1-2)}\cos\theta_4, \quad \mathrm{B}_3 = 0\,,
\end{aligned} \tag{4.17}$$

where $R_o \in \mathbb{R}^+$; $U_o \in \mathbb{R}$. Here α_1 and α_2 lie in: $1/2 < \alpha_1 < 2$ and $(4\alpha_1+1)/3 < \alpha_2 < 2\alpha_1$, $\alpha_1 > 2$ and $\alpha_1 < \alpha_2 < (4\alpha_1+1)/3$. The functions W_4 and θ_4 are given by

$$W_4 = \frac{(4\alpha_1 - 2)}{3\alpha_1}\left[\frac{\ln[t]}{\alpha_1} + z\right] + C_2, \quad \theta_4 = \theta_o + \frac{\ln[t]}{\alpha_1} + \frac{3}{(2-4\alpha_1)}\ln[W_4], \quad C_2,\, \theta_o \in \mathbb{R}\,.$$

Solutions (4.13)–(4.17) are real and non-singular if $W_i > 0$ $(i = 1,2,3,4)$ and $t > 0$, they all tend asymptotically to zero for $t \to \infty$. These solutions describe a nonstationary and compressible flow with vorticity that lies in the x-y plane. Note that for the solution (4.14) there is no gradient of fluid pressure. Except for the solution (4.13) where $\mathbf{F_m} = \mathbf{0}$, applying the expression for the Lorentz force (2.8) we find than that only the first term (i.e., the magnetic pressure) yields a nonzero contribution, while the second term (the magnetic tension force) vanishes. So, $\mathbf{F_m}$ is a conservative force that acts only along the z-axis as explained for solutions G_1 and G_2. Consequently, by virtue of Kelvin's theorem, the circulation of the fluid is constant for solutions (4.13)–(4.17).

5 $\mathbf{B} = \big(\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}\big)$

$\boldsymbol{G_4 = \{F + \alpha_1 G + \alpha_2 H,\ P_o,\ P_3\}}, \qquad \alpha_1 \neq 0, 1, \quad \alpha_2 \in \mathbb{R}$

By choosing $\alpha_2 = -1$, we were able to solve completely the reduced system and we get

$$\begin{aligned}
&\rho = x^{-2(1+\alpha_1)} R\,(s), \qquad p = A_o - \frac{(c_o^2 Y_o^2 + Z_o^2)}{2(x^2+y^2)}\csc^2\big(c_o\theta\big)\,,\\
&\mathrm{v}_1 = 0, \qquad \mathrm{v}_2 = 0, \qquad \mathrm{v}_3 = W_o\,(x^2+y^2)^{\alpha_1/2}\sin^{\alpha_1}\big(c_o\theta\big)\,,\\
&\mathrm{B}_1 = \frac{c_o Y_o\, x}{(x^2+y^2)}\cot(c_o\theta) + \frac{Y_o\, y}{(x^2+y^2)}\,,\\
&\mathrm{B}_2 = \frac{c_o Y_o\, y}{(x^2+y^2)}\cot\big(c_o\theta\big) - \frac{Y_o\, x}{(x^2+y^2)}, \qquad \mathrm{B}_3 = \frac{Z_o}{\sqrt{x^2+y^2}}\csc\big(c_o\theta\big)\,,
\end{aligned} \tag{5.1}$$

where $A_o \in \mathbb{R}^+$; c_o, W_o, Y_o, $Z_o \in \mathbb{R}$, with $c_o \neq 0$. As a result of the incompressibility of the fluid, R is an arbitrary function (but positively defined) of the symmetry variable $s = x/y$. The function θ is given by

$$\theta = \arctan\left(\frac{y}{x}\right). \tag{5.2}$$

Notice that for a given $c_o \in \mathbb{N}/\{0\}$, the trigonometric functions can be expanded in terms of rational functions of x and y. This solution is non-singular if $x, y \neq 0$, and $y \neq x\tan\big[k\pi/c_o\big]$, $k \in \mathbb{Z}$. We note that this solution does not depend on the variables z and t: the axis of symmetry coincides with the z-axis which is also the direction of the stationary flow (the paths of fluid elements are streamlines). Solution (5.1) describes a static equilibrium [c.f., the steady-state form of (2.2)]:

$$\nabla p = \mathbf{J} \times \mathbf{B}\,, \tag{5.3}$$

where the current density $\mathbf{J} = (\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3)$, given by the Ampère's law, takes the form

$$\mathrm{J}_1 = -\frac{\mathrm{B}_1}{Y_o}\mathrm{B}_3, \qquad \mathrm{J}_2 = -\frac{\mathrm{B}_2}{Y_o}\mathrm{B}_3, \qquad \mathrm{J}_3 = \frac{c_o^2\, Y_o}{(x^2+y^2)}\csc^2\big(c_o\theta\big)\,. \tag{5.4}$$

Nothing that $\mathbf{J} \times \mathbf{B} = -\nabla(|\mathbf{B}|^2/2) + (\mathbf{B} \cdot \nabla)\mathbf{B}$, so to achieve equilibrium the magnetic tension must balance not only the magnetic pressure gradient but that of the fluid pressure as well. Also, we have $(\mathbf{B} \cdot \nabla)\mathbf{v} = \mathbf{0}$ which indicates that the flow velocity $\mathbf{v}$ remains constant along $\mathbf{B}$. The flow has vorticity given by $\boldsymbol{\omega} = (\omega_1, \omega_2, 0)$, with

$$\omega_1 = \alpha_1 \frac{\big[x + c_o y \cot(c_o\theta)\big]}{(x^2 + y^2)} \, \mathrm{v}_3, \qquad \omega_2 = \alpha_1 \frac{\big[y - c_o x \cot(c_o\theta)\big]}{(x^2 + y^2)} \, \mathrm{v}_3 \, . \tag{5.5}$$

By virtue of Kelvin's theorem the circulation of the fluid is preserved because $\mathrm{d}\mathbf{v}/\mathrm{d}t = \mathbf{0}$ [c.f., equation (2.11) is identically satisfied].

The solution (5.1) can be interpreted physically as the propagation of a stationary double entropic wave $\mathrm{E}_1\mathrm{E}_1$ (resulting from the nonlinear superposition of two entropic simple waves E_1) in an incompressible fluid [12]. In the next two solutions, we present such type of wave in the context of cylindrically and spirally magnetic geometry.

$\boldsymbol{G_5 = \big\{J_3 + \alpha_1 G + \alpha_2 H,\ P_o,\ P_3\big\}}, \qquad \alpha_1,\ \alpha_2 \in \mathbb{R}$

From this algebra, we get the following invariant solution in cylindrical coordinates (r, φ, z)

$$\begin{aligned}
&\rho = \exp[2(\alpha_2 - \alpha_1)\varphi] R(r), \qquad p = A_o \exp\big[2\alpha_2\varphi - 2\alpha_2\vartheta(r)\big] \, , \\
&\mathrm{v}_r = 0, \qquad \mathrm{v}_\varphi = 0, \qquad \mathrm{v}_z = W_o \exp\big[\alpha_1\varphi - \alpha_2\vartheta(r)\big] \, , \\
&\mathrm{B}_r = \frac{X_o}{r}\exp\big[\alpha_2\varphi - \alpha_2\vartheta(r)\big] \, , \\
&\mathrm{B}_\varphi = \frac{X_o}{r}\exp\big[\alpha_2\varphi - \alpha_2\vartheta(r)\big] \tan\big[Y(r)\big], \quad \mathrm{B}_z = Z_o \exp\big[\alpha_2\varphi - \alpha_2\vartheta(r)\big] \, ,
\end{aligned} \tag{5.6}$$

where $A_o \in \mathbb{R}^+$, W_o, X_o, $Z_o \in \mathbb{R}$; R is an arbitrary function (with $R > 0$) of the symmetry variable $s = r$ (its level surfaces represent cylinders), and the function ϑ is given by

$$\vartheta = \int^r \tan(Y) \frac{\mathrm{d}r'}{r'} \, . \tag{5.7}$$

The function Y is a solution of the nonlinear first order ODE

$$r\frac{\mathrm{d}Y}{\mathrm{d}r} - \alpha_2\beta_o r^2 \cos^2(Y) - \alpha_2 = 0 \, , \tag{5.8}$$

where the parameter β_o is defined by

$$\beta_o = \frac{(2A_o + Z_o^2)}{X_o^2} \, . \tag{5.9}$$

Equation (5.8) does not have the Painlevé property, and is difficult to integrate. If such an equation has the Painlevé property, i.e., if its general solution has no "movable" singularities other than poles (such as essential singularities or branch points), then it can be transformed into one of transcendents forms (Ince [14]) and integrated in terms of some known functions. The test verifying whether a given ODE satisfies certain necessary condition for having the Painlevé property, is algorithmic [15] and can be performed using a specifically written MATHEMATICA program [16]. A complete analysis of this ODE

would take us beyond the scope of this paper. However, we can discuss some interesting physical properties of this solution.

The Ampère's law shows that three components of current, J_r, J_φ and J_z, are nonzero:

$$\mathrm{J}_r = \frac{\alpha_2}{r}\mathrm{B}_z, \quad \mathrm{J}_\varphi = \frac{\alpha_2}{r}\tan(Y)\mathrm{B}_z, \quad \mathrm{J}_z = \frac{\mathrm{B}_r}{\cos^2(Y)}\frac{\mathrm{d}Y}{\mathrm{d}r} - \frac{\alpha_2}{r}\mathrm{B}_\varphi\tan(Y) - \frac{\alpha_2}{r}\mathrm{B}_r\,. \quad (5.10)$$

For $\alpha_2 = 0$, the current density $\mathbf{J}$ is identically zero. So $\nabla\times\mathbf{B} = \mathbf{0}$ permits the expression of the magnetic field as the gradient of a magnetic scalar potential, $\mathbf{B} = -\nabla\phi_M$.

Notice that the equation (5.8) corresponds to the reduced form of the steady-state of equation (2.2) along both the r-and φ-axes:

$$\frac{\partial}{\partial r}\left[p + |\mathbf{B}|^2/2\right] = (\mathbf{B}\cdot\nabla)_r\,\mathrm{B}_r, \qquad \frac{1}{r}\frac{\partial}{\partial\varphi}\left[p + |\mathbf{B}|^2/2\right] = (\mathbf{B}\cdot\nabla)_\varphi\,\mathrm{B}_\varphi\,, \qquad (5.11)$$

which both describe a balance between the pressure gradient ∇p and the Lorentz force $\mathbf{F_m}$. The two terms on the left side of the equalities represent the fluid pressure and the magnetic pressure, while the terms on the right side represent the tension forces generated by the curvature of the magnetic field lines. Both the magnetic field $\mathbf{B}$ and current density $\mathbf{J}$ lie on constant pressure surfaces since $\mathbf{B}\cdot\nabla p = 0$ and $\mathbf{J}\cdot\nabla p = 0$.

Solution G_5 is non-singular when $Y \neq (2k+1)\pi/2$, $k\in\mathbb{Z}$, and $r \neq 0$. This solution is independent of z and t. The solution G_5 represents an equilibrium two-dimensional configuration for which the stationary and incompressible flow is along the symmetric axis (i.e., the z-axis) of a cylindrically symmetric magnetic field $\mathbf{B}$. Also, $(\mathbf{B}\cdot\nabla)\mathbf{v} = \mathbf{0}$ confirms that $\mathbf{v}$ remains constant along $\mathbf{B}$. The flow has a vorticity field given by

$$\boldsymbol{\omega} = \left(\frac{\alpha_1}{r}\mathrm{v}_z,\, \frac{\alpha_1}{r}\tan(Y)\mathrm{v}_z,\, 0\right)\,, \qquad (5.12)$$

and the circulation of the fluid is preserved since $\mathrm{d}\mathbf{v}/\mathrm{d}t = \mathbf{0}$.

Finally, if we impose the condition that

$$\mathbf{B}\cdot\nabla\rho = 0\,, \qquad (5.13)$$

then we can determine the explicit form of the density:

$$\rho = R_o\exp\left[2(\alpha_2-\alpha_1)\varphi - 2\alpha_2\vartheta(r)\right], \qquad R_o\in\mathbb{R}^+\,. \qquad (5.14)$$

Equation (5.13) simply states that the lines of force lie on the surfaces $\rho = \mathrm{const}$ which coincide with the isobaric surfaces. So, any given smooth equilibrium function $\mathbf{B}$, $\mathbf{v}$, ρ and p in $\mathbb{R}^3$ defines a distribution of magnetic surfaces $\psi = \mathrm{const}$ in $\mathbb{R}^3$ [17].

$$\boldsymbol{G_6} = \big\{\boldsymbol{J_3} + \boldsymbol{\alpha_1}(\boldsymbol{F}+\boldsymbol{G}) + \boldsymbol{\alpha_2 H},\ \boldsymbol{P_o},\ \boldsymbol{P_3}\big\}, \qquad \alpha_1 \neq 0,\ \ \alpha_2 \in \mathbb{R}$$

In cylindrical coordinates (r, φ, z), the corresponding GIS to this algebra has the form

$$\begin{aligned}
&\rho = \exp\big[2(\alpha_2-\alpha_1)\varphi\big]\, R(s), \qquad p = A_o \exp\big[2\alpha_2\varphi - 2\alpha_2\vartheta_2(s)\big]\,, \\
&\mathrm{v}_r = 0, \qquad \mathrm{v}_\varphi = 0, \qquad \mathrm{v}_z = W_o \exp\big[\alpha_1\varphi - \alpha_2\vartheta_2(s)\big]\,, \\
&\mathrm{B}_r = X_o \frac{\exp\big[\alpha_2\varphi - \vartheta_1(s) - \alpha_2\vartheta_2(s)\big]}{\cos[Y(s)] - \alpha_1\sin[Y(s)]}\cos[Y(s)]\,, \\
&\mathrm{B}_\varphi = X_o \frac{\exp\big[\alpha_2\varphi - \vartheta_1(s) - \alpha_2\vartheta_2(s)\big]}{\cos[Y(s)] - \alpha_1\sin[Y(s)]}\sin[Y(s)], \qquad \mathrm{B}_z = Z_o\big[\alpha_2\varphi - \alpha_2\vartheta_2(s)\big]\,,
\end{aligned} \tag{5.15}$$

where $A_o \in \mathbb{R}^+$, W_o, X_o, $Z_o \in \mathbb{R}$; and R is an arbitrary function (but positively defined) of the symmetry variable $s = re^{-\alpha_1\varphi}$. The level surfaces of the symmetry variables are logarithmic spirals. The functions ϑ_1 and ϑ_2 are given by

$$\vartheta_1 = \int^s \frac{\cos(Y)}{\big[\cos(Y) - \alpha_1\sin(Y)\big]}\frac{\mathrm{d}s'}{s'}, \qquad \vartheta_2 = \int^s \frac{\sin(Y)}{\big[\cos(Y) - \alpha_1\sin(Y)\big]}\frac{\mathrm{d}s'}{s'}\,. \tag{5.16}$$

The functions Y solves the integro-differential equation:

$$s\frac{\mathrm{d}Y}{\mathrm{d}s} - \frac{\alpha_2\beta_o}{(1+\alpha_1^2)}\,[\cos(Y) - \alpha_1\sin(Y)]^2 \exp\,[2\vartheta_1\,(s)] - \frac{(\alpha_1+\alpha_2)}{(1+\alpha_1^2)} = 0\,, \tag{5.17}$$

where β_o retains the form (5.9), and with the conditions that $Y \neq \arctan(1/\alpha_1) + k\pi$, $k \in \mathbb{Z}$, and $s \neq 0$, which ensure that the solution G_6 is non-singular. This solution does not depend on the variables z and t. Equation (5.17) corresponds to the same reduced form of $\nabla p = \mathbf{J}\times\mathbf{B}$ along both the r- and φ-axes [c.f., equation (5.11)]. The determination of the general solution of equation (5.17) is a difficult task. We have not able to find an explicit solution of (5.17) and we must look for a solution by numerical methods. Nevertheless, we can resume some physical properties of this solution.

The solution G_6 represents a two-dimensional spirally symmetric configuration in static equilibrium; the steady flow is parallel the axis of symmetry (i.e., the z-axis), and is conserved along the magnetic field lines since $(\mathbf{B}\cdot\nabla)\mathbf{v} = \mathbf{0}$. The vorticity of the flow is given by $\boldsymbol{\omega} = (\omega_r, \omega_\varphi, 0)$, with

$$\omega_r = \frac{\alpha_1}{r}\frac{\cos(Y)\mathrm{v}_z}{\big[\cos(Y) - \alpha_1\sin(Y)\big]}\,, \qquad \omega_\varphi = \frac{\alpha_1}{r}\frac{\sin(Y)\mathrm{v}_z}{\big[\cos(Y) - \alpha_1\sin(Y)\big]}\,, \tag{5.18}$$

and the circulation of the fluid is preserved since we have $\mathrm{d}\mathbf{v}/\mathrm{d}t = \mathbf{0}$.

The current induced by $\mathbf{B}$ has three nonzero components J_r, J_φ and J_z, given by:

$$\begin{aligned}
\mathrm{J}_r &= \frac{\alpha_2}{r}\frac{\cos(Y)\mathrm{B}_z}{[\cos(Y) - \alpha_1\sin(Y)]}, \qquad \mathrm{J}_\varphi = \frac{\alpha_2}{r}\frac{\sin(Y)\mathrm{B}_z}{[\cos(Y) - \alpha_1\sin(Y)]}\,, \\
\mathrm{J}_z &= \frac{1}{r}\big(\mathrm{B}_\varphi - \alpha_2\mathrm{B}_r\big) - \frac{\big[\cos(Y) + \alpha_2\sin(Y)\big]}{r\big[\cos(Y) - \alpha_1\sin(Y)\big]}\big(\mathrm{B}_\varphi + \alpha_1\mathrm{B}_r\big) \\
&\quad + (1+\alpha_1^2)\exp[-\alpha_1\varphi]\,\frac{\big[\cos(Y)\mathrm{B}_r + \sin(Y)\mathrm{B}_\varphi\big]}{\big[\cos(Y) - \alpha_1\sin(Y)\big]}\frac{\mathrm{d}Y}{\mathrm{d}s}\,.
\end{aligned} \tag{5.19}$$

If we impose the condition that $\mathbf{B} \cdot \nabla \rho = 0$, then we get the expression for ρ

$$\rho = R_o \exp\big[2(\alpha_2 - \alpha_1)\varphi - 2\alpha_2\vartheta_2(s)\big], \qquad R_o \in \mathbb{R}^+ . \tag{5.20}$$

So $\mathbf{B}$, $\mathbf{v}$, ρ and p define now a distribution of magnetic surfaces $\psi = \text{const}$ in $\mathbb{R}^3$.

It is noteworthy that for $\alpha_2 = 0$, the equation (5.17) can be easily solved to give:

$$\begin{aligned}
&\rho = \exp\big[-2\alpha_1\varphi\big] R(s), \qquad p = A_o\,; \qquad \mathrm{v}_r = 0, \qquad \mathrm{v}_\varphi = 0\,, \\
&v_z = W_o\, r^{\alpha_1/(\alpha_1^2+1)} \exp\left[\frac{\alpha_1}{(\alpha_1^2+1)}\,\varphi\right]\big[\alpha_1 \sin(Y) - \cos(Y)\big]\,, \\
&\mathrm{B}_r = X_o\, r^{-1/(\alpha_1^2+1)} \exp\left[\frac{\alpha_1}{(\alpha_1^2+1)}\,\varphi\right]\cos(Y)\,, \\
&\mathrm{B}_\varphi = X_o\, r^{-1/(\alpha_1^2+1)} \exp\left[\frac{\alpha_1}{(\alpha_1^2+1)}\,\varphi\right]\sin(Y)\,, \qquad \mathrm{B}_z = Z_o\,,
\end{aligned} \tag{5.21}$$

where $A_o \in \mathbb{R}^+$, $\alpha_1 \neq 0$, W_o, X_o, $Z_o \in \mathbb{R}$; and R is an arbitrary function (with $R > 0$) of $s = re^{-\alpha_1\varphi}$. The expression for the function Y is given by

$$Y = Y_o + \frac{\alpha_1}{(\alpha_1^2+1)}\ln[r] - \frac{\alpha_1^2}{(\alpha_1^2+1)}\,\varphi\,. \tag{5.22}$$

Here we have an example of a two-dimensional configuration with spirally symmetric magnetic geometry. Since $\mathbf{J} = \mathbf{0}$, the magnetic field $\mathbf{B}$ is potential.

$\boldsymbol{G_7 = \{J_3 + P_3,\ P_1,\ P_2\}}$

The reductions of this algebra lead us to this invariant solution of physical interest:

$$\begin{aligned}
&\rho = \rho_o, \qquad p = p_o\,, \\
&\mathrm{v}_1 = \frac{U(t)}{\sqrt{\rho_o}}\sin\big[V(t) - z\big], \qquad \mathrm{v}_2 = \frac{U(t)}{\sqrt{\rho_o}}\cos\big[V(t) - z\big], \qquad \mathrm{v}_3 = W_o\,, \\
&\mathrm{B}_1 = X(t)\sin\big[Y(t) - z\big], \qquad \mathrm{B}_2 = X(t)\cos\big[Y(t) - z\big], \qquad \mathrm{B}_3 = Z_o\,,
\end{aligned} \tag{5.23}$$

where ρ_o, $p_o \in \mathbb{R}^+$; W_o, $Z_o \in \mathbb{R}$ and $Z_o \neq 0$. The functions U, V, X and Y have the form

$$\begin{aligned}
&U(t) = \left[E_o - \sqrt{E_o^2 - \rho_o\delta_o^2}\,\sin\left(\phi_o - \frac{2Z_o}{\sqrt{\rho_o}}t\right)\right]^{1/2}, \\
&X(t) = \left[E_o + \sqrt{E_o^2 - \rho_o\delta_o^2}\,\sin\left(\phi_o - \frac{2Z_o}{\sqrt{\rho_o}}t\right)\right]^{1/2}, \\
&V(t) = V_o + W_o\,t + \arctan\left[\frac{E_o}{\sqrt{\rho_o}\,\delta_o}\tan\left(\frac{\phi_o}{2} - \frac{Z_o}{\sqrt{\rho_o}}t\right) - \frac{\sqrt{E_o^2 - \rho_o\delta_o^2}}{\sqrt{\rho_o}\,\delta_o}\right], \\
&Y(t) = V + \arccos\left[\frac{\sqrt{\rho_o}\,\delta_o}{UX}\right],
\end{aligned}$$

where ϕ_o, V_o and $Y_o \in \mathbb{R}$. The constant E_o is equal to

$$E_o = \frac{\rho_o}{2}[U(t)]^2 + \frac{1}{2}[X(t)]^2\,, \tag{5.24}$$

which corresponds to the sum of kinetic and magnetic energies perpendicular to the z-axis. The constant δ_o is given by

$$\delta_o = \mathrm{v}_1 \mathrm{B}_1 + \mathrm{v}_2 \mathrm{B}_2 \,, \tag{5.25}$$

with $\delta_o^2 \leq E_o^2/\rho_o$ ensuring that the solution G_7 is always real. Thus, the longitudinal component of $\mathbf{v}$ along the magnetic field $\mathbf{B}$ is a constant quantity.

The solution (5.23) describes, along with the relations (5.24) and (5.25), the main properties of the double Alfvén-entropic wave AE_1 which results from the nonlinear superposition of an Alfvén wave A with an entropic wave E_1 [12]. Such waves of an arbitrary amplitude were found for the first time by Alfvén [1] as nonstationary solutions of MHD equations for an incompressible medium (producing no density or pressure fluctuations). The wave motion can be explained by the intrinsic nature of the Lorentz force

$$\mathbf{F_m} = (\mathbf{B}\cdot\nabla)\mathbf{B} = -Z_o \mathrm{B}_2\, \mathbf{e}_1 + Z_o \mathrm{B}_1\, \mathbf{e}_2 \,, \tag{5.26}$$

which corresponds to tension force along the field lines of $\mathbf{B}$. So that the magnetic field lines twist relatively to one another, but do not compress. Namely, the double wave AE_1 modifies the flow direction that is not parallel to the magnetic field $\mathbf{B}$. The flow has vorticity that lies in the x-y plane, and according to the Kelvin's theorem, the circulation of the fluid is not conserved because of the presence of tension forces originating from $\mathbf{F_m}$. Note that the instantaneous power dissipated by the magnetic force is given by

$$\mathbf{F_m}\cdot\mathbf{v} = \frac{\sqrt{E_o^2 - \rho_o \delta_o^2}}{\sqrt{\rho_o}} \cos\left(\phi_o - \frac{2Z_o}{\sqrt{\rho_o}} t\right) \,. \tag{5.27}$$

Indeed, the equation (5.24) describes basic oscillation between perpendicular fluid kinetic energy and perpendicular "line bending" magnetic energy, i.e., a balance between inertia and field lines tension. For the particular case $E_o = \sqrt{\rho_o}|\delta_o|$; $\mathbf{v}$ are $\mathbf{B}$ collinear: there is no coupling between flow and $\mathbf{B}$. Finally, note that the conservation law of energy [2]

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho|\mathbf{v}|^2 + \frac{p}{(\gamma-1)} + \frac{|\mathbf{B}|^2}{2}\right] + \nabla\cdot\left\{\left[\frac{1}{2}\rho|\mathbf{v}|^2 + \frac{\gamma p}{(\gamma-1)} + |\mathbf{B}|^2\right]\mathbf{v} + (\mathbf{B}\cdot\mathbf{v})\mathbf{B}\right\} = 0\,,$$

turns out to be trivially satisfied.

$$\boldsymbol{G_8} = \{\boldsymbol{F}+\boldsymbol{G},\ \boldsymbol{K_1}+\boldsymbol{P_2}+\boldsymbol{\alpha P_3},\ \boldsymbol{P_1}+\boldsymbol{\beta P_2}\}, \qquad \alpha\in\mathbb{R}, \quad \beta>0$$

The corresponding invariant solution has the explicit form

$$\begin{aligned}
&\rho = \frac{R(t)}{[\alpha(\beta x - y) + (1-\beta t)z]^2}, \qquad p = A_o\big[R(t)\big]^{-\gamma}\,, \\
&\mathrm{v}_1 = \frac{z}{\alpha} + \frac{(W_o\, t - U_o)}{\alpha R(t)}\big[\alpha(\beta x - y) + (1-\beta t)z\big]\,, \\
&\mathrm{v}_2 = \frac{V_o}{R(t)}\big[\alpha(\beta x - y) + (1-\beta t)z\big], \quad \mathrm{v}_3 = -\frac{W_o}{R(t)}\big[\alpha(\beta x - y) + (1-\beta t)z\big]\,, \\
&\mathrm{B}_1 = \frac{[Z_o\, t + \alpha X_o]}{\alpha R(t)}, \qquad \mathrm{B}_2 = \frac{Y_o}{R(t)}, \qquad \mathrm{B}_3 = \frac{Z_o}{R(t)}\,,
\end{aligned} \tag{5.28}$$

where A_o, $W_o \in \mathbb{R}^+$; U_o, V_o, X_o, Y_o, $Z_o \in \mathbb{R}$, $\alpha \in \mathbb{R}/\{0\}$. The function $R(t)$ is given by

$$R(t) = \beta W_o\, t^2 - [W_o + \beta U_o + \alpha V_o]t + R_o\,, \tag{5.29}$$

where the constant R_o must satisfy

$$R_o > \frac{[W_o + \beta U_o + \alpha V_o]^2}{2\beta W_o}\,, \tag{5.30}$$

in order to obtain $\rho > 0$. This solution represents a nonstationary and compressible flow in $(3+1)$ dimensions. Note that this solution is singular when $t = [z + \alpha(\beta x - y)]/\beta z$ which coincides with a stagnation point in $\mathbb{R}^4$ (i.e., where $\rho \to \infty$, $\mathbf{v} = \mathbf{0}$), and tends asymptotically to zero for sufficiently large t. The fluid is force-free since both the gradient of fluid pressure and $\mathbf{F_m}$ are zero. Force-free conditions are widely applicable in astrophysical environments because forces other than electromagnetic are comparatively much smaller [18]. The vorticity field of the flow has the form $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, with

$$\omega_1 = \frac{[\beta V_o\, t + \alpha W_o - V_o]}{R(t)}, \quad \omega_2 = \frac{[R_o - U_o + \alpha^2\beta W_o - \alpha V_o t]}{\alpha R(t)}, \quad \omega_3 = \frac{[W_o t + \alpha\beta V_o - U_o]}{R(t)}\,,$$

Consequently, by virtue of Kelvin's theorem, the circulation of the fluid is preserved.

$\boldsymbol{G_9 = \{F + K_3 + \alpha H,\ P_2,\ P_3 + \beta P_1\}}$, $\quad \beta > 0, \quad \alpha \in \mathbb{R}$

The reductions of this algebra lead us to the corresponding analytical solution:

$$\begin{aligned}
&\rho = \frac{R_o}{f} t^{2\alpha} \exp\left[-(1+2\alpha)\int^s \frac{\mathrm{d}s'}{f}\right], \qquad p = \frac{A_o}{f^\gamma} t^{2\alpha} \exp\left[-(2\alpha+\gamma)\int^s \frac{\mathrm{d}s'}{f}\right], \\
&\mathrm{v}_1 = U(s), \qquad \mathrm{v}_2 = V_o, \qquad \mathrm{v}_3 = \ln[t] + W(s), \\
&\mathrm{B}_1 = \beta\frac{Z_o}{f} t^{\alpha} \exp\left[-(1+\alpha)\int^s \frac{\mathrm{d}s'}{f}\right], \qquad \mathrm{B}_2 = \frac{Y_o}{f} t^{\alpha} \exp\left[-(1+\alpha)\int^s \frac{\mathrm{d}s'}{f}\right], \\
&\mathrm{B}_3 = \frac{Z_o}{f} t^{\alpha} \exp\left[-(1+\alpha)\int^s \frac{\mathrm{d}s'}{f}\right],
\end{aligned} \tag{5.31}$$

where R_o, A_o, V_o, Y_o and Z_o are arbitrary constants. Also, we have introduced the function

$$f = U(s) - \beta W(s) - s + \beta\,, \tag{5.32}$$

that depends on the symmetry variable $s = \beta \ln[t] + (x - \beta z)/t$. The unknown functions U and W are obtained by solving the overdetermined system which consists of (5.32) and these two equations:

$$\beta\frac{\mathrm{d}U}{\mathrm{d}s} + \frac{\mathrm{d}W}{\mathrm{d}s} + 1 = 0\,, \tag{5.33}$$

$$\begin{aligned}
f^3\frac{\mathrm{d}f}{\mathrm{d}s} + f^3 - \beta f^2 - (1+\beta^2)\frac{A_o}{R_o}\left[\gamma\frac{\mathrm{d}f}{\mathrm{d}s} + \gamma + 2\alpha\right] f^{2-\gamma} \exp\left[(1-\gamma)\int^s \frac{\mathrm{d}s'}{f}\right] \\
- \frac{(1+\beta^2)}{R_o}\left[Y_o^2 + (1+\beta^2)Z_o^2\right]\left[\frac{\mathrm{d}f}{\mathrm{d}s} + 1 + \alpha\right] \exp\left[-\int^s \frac{\mathrm{d}s'}{f}\right] = 0\,.
\end{aligned} \tag{5.34}$$

We have not succeeded in solving completely this system. However, with the ansatz

$$W = C_1 s + C_2, \qquad C_1,\ C_2 \in \mathbb{R}, \qquad C_1 \neq 0\,, \tag{5.35}$$

we can calculate f from equation (5.34) by setting $C_1 = -1/2$ and $\gamma = 3$. Next, by substituting this result into equations (5.32) and (5.33), we determine U and W. Finally, we obtain the following invariant solution of Eqs (2.1)–(2.5):

$$\begin{aligned}
\rho &= -\frac{1}{2\beta}(2\alpha+1)(1+\beta^2)\big[Y_o^2 + (1+\beta^2)Z_o^2\big]t^{2\alpha}\,\xi^{(4\alpha+1)}\,,\\
p &= -\frac{1}{2\beta}\frac{(2\alpha+1)}{(4\alpha+3)}\big[Y_o^2 + (1+\beta^2)Z_o^2\big]t^{2\alpha}\,\xi^{(4\alpha+3)}\,,\\
\mathrm{v}_1 &= \frac{2\beta}{(1+\beta^2)}\ln\left[t^{1/4}\,\xi\right] + \frac{(x-\beta z)}{2(1+\beta^2)t} + \frac{C_2+\beta(C_3-1)}{(1+\beta^2)}, \qquad \mathrm{v}_2 = V_o\,,\\
\mathrm{v}_3 &= \frac{2}{(1+\beta^2)}\ln\left[t^{(2+\beta^2)/4}\,\xi\right] + \frac{\beta(\beta z-x)}{2(1+\beta^2)t} + \frac{C_3+\beta(\beta-C_2)}{(1+\beta^2)}\,,\\
\mathrm{B}_1 &= \beta Z_o\,t^{2\alpha}\,\xi^{(2\alpha+1)}, \qquad \mathrm{B}_2 = Y_o\,t^{2\alpha}\,\xi^{(2\alpha+1)}, \qquad \mathrm{B}_3 = Z_o\,t^{2\alpha}\,\xi^{(2\alpha+1)}\,,
\end{aligned} \tag{5.36}$$

where

$$\xi = C_2 - \frac{\beta}{2}\ln[t] - \frac{(x-\beta z)}{2t}\,, \tag{5.37}$$

and $\beta > 0$, C_2, C_3, V_o, Y_o and $Z_o \in \mathbb{R}$. Solution (5.36) is always real and non singular if $\xi > 0$ and $t > 0$. Note that this solution does not depend on the variable y. In order that ρ and $p > 0$, we must have $-3/4 < \alpha < -1/2$. This solution describes a nonstationary and compressible flow in $(2+1)$ dimensions with vorticity along the axis of symmetry:

$$\boldsymbol{\omega} = \big[t\xi\big]^{-1}\,\mathbf{e}_2\,, \tag{5.38}$$

which is everywhere normal to the plane of flow (the x-z plane). The current density induced by $\mathbf{B}$ takes the form

$$\mathbf{J} = -(2\alpha+1)Z_o\,t^{(\alpha-1)}\,\xi^{2\alpha}\Big(1, -(1+\beta^2), 1\Big)\,. \tag{5.39}$$

Thus, the Lorentz force related to solution (5.36) is given by

$$\mathbf{F_m} = \frac{\beta}{(1+\beta^2)}\frac{\rho}{t}\Big(1,\ 0,\ -\beta\Big) \tag{5.40}$$

and produced no tension along the lines of force. Also, we have $(\mathbf{B}\cdot\nabla)\mathbf{v} = \mathbf{0}$ which implies that $\mathbf{B}/\rho$ is constant along the streamlines, and thus $\mathbf{B}$ remains inextensible [c.f., (2.10)]. So, along both the x- and z-axes the Lorentz force $\mathbf{F_m}$ causes compressions and expansions of the distance between the lines of force without changing their direction, exactly as does the propagation of a double magnetoacoustic wave FF which results from the nonlinear superposition of two magnetoacoustic fast waves F that propagate perpendicular to each other in an ideal fluid [12]. Since $\mathbf{F_m}$ is a conservative force (i.e., it comes from the gradient of magnetic pressure $|\mathbf{B}|^2/2$), the circulation of the fluid (4.3) is conserved.

$\boldsymbol{G_{10} = \{G + \alpha H, K_1, K_2\}}, \qquad \alpha \in \mathbb{R}$

For $\alpha = 2$, the reduced system corresponding to this algebra admits this exact solution

$$
\begin{aligned}
&\rho = \frac{R_o}{t^2} W^{-1}, \qquad p = \frac{A_o}{t^4} W^{-\gamma} \exp\left[2(2-\gamma)\int^z \frac{\mathrm{d}z'}{W}\right], \\
&\mathrm{v}_1 = \frac{x - U_o}{t} + \frac{a_o^2}{Z_o}\frac{X_o\, t^{-1}}{(W - a_o^2)} \exp\left[\int^z \frac{\mathrm{d}z'}{(W - a_o^2)}\right], \\
&\mathrm{v}_2 = \frac{y - V_o}{t} + \frac{a_o^2}{Z_o}\frac{Y_o\, t^{-1}}{(W - a_o^2)} \exp\left[\int^z \frac{\mathrm{d}z'}{(W - a_o^2)}\right], \qquad \mathrm{v}_3 = \frac{W}{t}, \\
&\mathrm{B}_1 = \frac{X_o\, t^{-2}}{(W - a_o^2)} \exp\left[\int^z \frac{\mathrm{d}z'}{(W - a_o^2)}\right], \quad \mathrm{B}_2 = \frac{Y_o\, t^{-2}}{(W - a_o^2)} \exp\left[\int^z \frac{\mathrm{d}z'}{(W - a_o^2)}\right], \quad \mathrm{B}_3 = \frac{Z_o}{t^2},
\end{aligned}
\tag{5.41}
$$

where R_o, A_o, U_o, V_o, X_o, Y_o and $Z_o \in \mathbb{R}$; R_o and $Z_o \neq 0$, and $a_o^2 = Z_o^2/R_o$. The unknown function W of the symmetry variable $s = z$ is determined by solving the integro-differential equation which corresponds to reduced form of equation (2.2) along the z-axis:

$$
\begin{aligned}
W\frac{dW}{dz} - W + \frac{A_o}{R_o} W^{-\gamma}\left[2(2-\gamma) - \gamma\frac{\mathrm{d}W}{\mathrm{d}z}\right] \exp\left[2(2-\gamma)\int^z \frac{\mathrm{d}z'}{W}\right] \\
-\frac{(X_o^2 + Y_o^2)}{R_o}\frac{W}{(W - a_o^2)^3}\left[\frac{\mathrm{d}W}{\mathrm{d}z} - 1\right] \exp\left[2\int^z \frac{\mathrm{d}z'}{(W - a_o^2)}\right] = 0\,.
\end{aligned}
\tag{5.42}
$$

Equation (5.42) is difficult to solve completely. However, once again, with the ansatz

$$
W = C_1 z + C_2, \qquad C_1,\ C_2 \in \mathbb{R}, \qquad C_1 \neq 0\,,
$$

we can transform (5.42) into an algebraic equation:

$$
(C_1 - 1) + \frac{A_o}{R_o}(m+1)W^m - \frac{[X_o^2 + Y_o^2]}{R_o}(C_1 - 1)\big(W - a_o^2\big)^n = 0\,,
$$

where $m = [4 - C_1 + \gamma(C_1 + 2)]/C_1$ and $n = (2 - 3C_1)/C_1$. Then:

Case 1.) $C_1 = 1, \quad \gamma = 4/3$

$$
\begin{aligned}
&\rho = \frac{R_o}{t^2(z + C_2)}, \qquad p = \frac{A_o}{t^4}, \\
&\mathrm{v}_1 = \frac{(x - U_o)}{t} + \frac{Z_o}{t}\frac{X_o}{R_o}, \qquad \mathrm{v}_2 = \frac{(y - V_o)}{t} + \frac{Z_o}{t}\frac{Y_o}{R_o}, \qquad \mathrm{v}_3 = \frac{(z + C_2)}{t}, \\
&\mathrm{B}_1 = \frac{X_o}{t^2}, \qquad \mathrm{B}_2 = \frac{Y_o}{t^2}, \qquad \mathrm{B}_3 = \frac{Z_o}{t^2},
\end{aligned}
\tag{5.43}
$$

where $R_o \neq 0$, $A_o \in \mathbb{R}^+$; C_2, U_o, V_o, X_o, Y_o, $Z_o \in \mathbb{R}$, $Z_o \neq 0$. To avoid singularities and to obtain $\rho > 0$, we must have $t \neq 0$ and $z > -C_2$. Under these conditions, this solution tends asymptotically to zero when $t \to \infty$. It represents a compressible and irrotational flow of a force-free fluid.

Case 2.) $C_1 = 2/3, \quad \gamma = 5/4$

$$\begin{aligned}
\rho &= \frac{1}{t^2}\left[2A_o + (X_o^2+Y_o^2)\right]\left(\frac{2}{3}z + C_2\right)^{-1}, \qquad p = \frac{A_o}{t^4}\left(\frac{2}{3}z + C_2\right), \\
\mathrm{v}_1 &= \frac{(x-U_o)}{t} + \frac{a_o^2}{t}\frac{X_o}{Z_o}\left(\frac{2}{3}z + C_2 - a_o^2\right)^{1/2}, \\
\mathrm{v}_2 &= \frac{(y-V_o)}{t} + \frac{a_o^2}{t}\frac{Y_o}{Z_o}\left(\frac{2}{3}z + C_2 - a_o^2\right)^{1/2}, \qquad \mathrm{v}_3 = \frac{1}{t}\left(\frac{2}{3}z + C_2\right), \\
\mathrm{B}_1 &= \frac{X_o}{t^2}\left(\frac{2}{3}z + C_2 - a_o^2\right)^{1/2}, \qquad \mathrm{B}_2 = \frac{Y_o}{t^2}\left(\frac{2}{3}z + C_2 - a_o^2\right)^{1/2}, \qquad \mathrm{B}_3 = \frac{Z_o}{t^2},
\end{aligned} \tag{5.44}$$

where $A_o \in \mathbb{R}^+$; C_2, U_o, V_o, X_o, Y_o, $Z_o \in \mathbb{R}$, $Z_o \neq 0$. The parameter a_o^2 takes the form

$$a_o^2 = Z_o^2 \big/ \left[2A_o + (X_o^2+Y_o^2)\right],$$

which can be interpreted as the square of the Alfvén velocity. The solution (5.44) is real and non-singular if $z > 3(a_o^2 - C_2)/2$ and $t \neq 0$, and tends asymptotically to zero when $t \to \infty$. This solution describes the propagation of a compressional Alfvén wave in an ideal fluid [12],[19]. This can be illustrated with the expression of the Lorentz force:

$$\mathbf{F_m} = \frac{Z_o}{3t^4}\left(\frac{2}{3}z + C_2 - a_o^2\right)^{-1/2}\left[X_o\,\mathbf{e}_1 + Y_o\,\mathbf{e}_2\right] - \frac{(X_o^2+Y_o^2)}{3t^4}\,\mathbf{e}_3\,. \tag{5.45}$$

The first term on the right-hand side of this equation represents the action of a tension force along the magnetic field lines that acts only in the x-y plane (they twist relatively to one another, but do not compress), while the second term corresponds to the gradient of magnetic pressure, acting perpendicular to the x-y plane, which causes compressions and expansions of the distance between the lines of force without changing their direction. So, the solution (5.44) describes a basic oscillation between kinetic fluid energy (fluid inertia) and compressional (field line pressure) plus line bending (field line tension) magnetic energy. The vorticity of the flow is given by

$$\boldsymbol{\omega} = \frac{a_o^2}{3Z_o}t^{-1}\left(\frac{2}{3}z + C_2 - a_o^2\right)^{-1/2}\Big(-Y_o, X_o, 0\Big)\,. \tag{5.46}$$

However, by virtue of Kelvin's theorem, the circulation of the fluid is not conserved because of the presence of tension force originating from the Lorentz force: the term on the right-hand of the equation (2.11) is nonzero.

6 Concluding remarks

We summarize the main results achieved in this paper. Using the subgroup structure of the symmetry group of the MHD equations (2.1)-(2.5) to compute G-invariant solutions (GIS) of this system of PDEs, we have considered three-dimensional subgroups $\mathcal{G}$, where $\mathrm{codim}[\mathcal{G}] = 1$ in the space of independent variables E, in order to obtain reduced systems strictly composed of ODEs. The Lie algebras of these groups are representatives of the conjugacy classes of the Galilean-similitude (GS) algebra that have been calculated previously by Grundland and Lalague [9]. By means of a procedure of symmetry reduction of

Eqs (2.1)–(2.5), we have found some novel (at the best of our knowledge) exact solutions which demonstrate the efficiently of the SRM. Such solutions have been discussed by the point of view of their possible meaning.

In Section 4, we present solutions which describe a nonstationary and compressible flow in the presence of a magnetic field $\mathbf{B} = (\mathrm{B}_1, \mathrm{B}_2, 0)$. In particular, the Lorentz force (except for solution (4.13) where $\mathbf{F_m} = \mathbf{0}$) acts perpendicularly to $\mathbf{B}$, causing compressions and expansions of the lines of force without changing their direction, as does a magnetoacoustic fast wave F which propagates perpendicularly to $\mathbf{B}$ in an ideal fluid. There is no tension force, so the circulation of the fluid is conserved, which is also meaning that the vortex lines are frozen into the fluid.

In Section 5, we obtained several types of solutions where $\mathbf{B} = (\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3)$. For the algebras G_4, G_5 and G_6 we find stationary solutions for which the fluid is incompressible and its flow is along the axis of symmetry in the context of slab, cylindrically and spirally geometry, respectively. Solution G_7 describes the propagation of a nonstationary Alfvén wave in an incompressible fluid: $\mathbf{F_m}$ is purely a tension force and the circulation of the fluid is not conserved. Solution G_8 represents a nonstationary and compressive flow of a force-free fluid in $(3 + 1)$ dimensions. Solution G_9 describes the propagation of a double magnetoacoustic wave FF in a nonstationary and compressible fluid: $\mathbf{F_m}$ causes compressions and expansions of the lines of force along both the x- and z-axes. There is no tension force, so the circulation of the fluid is conserved. Finally for the algebra G_{10}, we obtain solutions (5.43) and (5.44). The first represents a compressible and irrotational flow of a force-free fluid. The second one corresponds to the propagation of a "compressional Alfvén wave" in an ideal fluid. Namely, it combines the action of tension force along the lines of force and the compressions and expansions of those resulting from magnetic pressure force. Obviously, the circulation of the fluid is not conserved.

It is suitable to recall than an approach of group-theoretical type to the equations for the fluid dynamics, especially to the isentropic compressible fluid model and to the MHD equations, is not new and has been carried out by many authors (see, for example, [8], [20]-[22]). On the other hand, a full classification of Lie subgroups of the symmetry group of MHD equations in $(3 + 1)$ dimensions was never been obtained before [9], and therefore allows us for a systematic approach to the task of constructing both invariant and partially invariant solutions of Eqs (2.1)–(2.5). The objective of future study is to take full advantage of some recently developed alternative versions of SRM, namely the method of partially invariant solutions (PIS) and the weak transversality method (WTM).

The notion of PIS was introduced by Ovsiannikov [3] and it relates to the case when only a part of the graph Γ_f of the solution $\mathrm{u} = f(\mathrm{x})$ is G-invariant. This means that a solution $\mathrm{u} = f(\mathrm{x})$ is a partially invariant solution if the number

$$\delta = \dim(\mathcal{G}\Gamma_f) - p\,, \tag{6.1}$$

(where $\mathcal{G}\Gamma_f$ is the orbit of the graph Γ_f), called the defect structure of the function $f(\mathrm{x})$ with respect to the group $\mathcal{G}$, satisfies the condition

$$0 < \delta < \min(r, q)\,, \tag{6.2}$$

where r is the dimension of the orbits of $\mathcal{G}$ in $E \times U$. The construction of this type of solution leads, through the reduction of the original system Δ, to two differential

systems of equations: Δ^1 for δ dependent and p independent variables and $\Delta/\mathcal{G}$ for $(q-\delta)$ dependent and $(p+\delta-r)$ independent variables. There is no longer a one-to-one correspondence between PIS of Δ and the solutions of $\Delta/\mathcal{G}$. For one solution of $\Delta/\mathcal{G}$ we have a family of solutions of Δ. In other words, this approach generates more solutions, covering much larger classes of initial and boundary conditions than the classical method. Nevertheless, there have been very few examples of its application. Construction of PIS requires a much more complex procedure than the "classical" SRM, but Grundland and Lalague [23] developed an effective algorithmic tool for this purpose.

The future study of MHD equations will focus on the construction of PIS invariant under four-dimensional subgroups (and some three-dimensional subgroups as mentioned above) with the defect structure $\delta = 1$. Under these assumptions the original equations can be reduced to the systems $\Delta/\mathcal{G}$ of ODEs, and one PDE denoted by Δ^1 for one unknown function. It could be mentioned also that the procedure cited above produces many new reducible solutions which would be, paradoxically, more difficult to obtain by the SRM, since they are related to lower dimensional subgroups than $\mathcal{G}$, leading to the reductions of Δ to PDEs rather than ODEs.

The second component of a future project will involve the recently WTM [24]. This new method is based on a "group invariant solutions without transversality" approach developed by Anderson *et al.* [25] which was designed to overcome the limitation of the SRM resulting from the transversality requirement, and extends the applicability of this classical method. The notion of transversality refers to the condition imposed on the vector fields generating subalgebras of a given system. Consider a r-dimensional subgroup $\mathcal{G}_o \subset \mathcal{G}$, or its corresponding subalgebra $\mathcal{L}_o \subset \mathcal{L}$, generated by vector fields

$$\hat{X}_a = \xi_a^{\mu}(\mathrm{x},\mathrm{u})\partial_{x^\mu} + \phi_a^j(\mathrm{x},\mathrm{u})\partial_{u^j}\,, \qquad a = 1,\,\ldots\,,r \tag{6.3}$$

with the matrix of characteristics of the vector fields $\hat{X}_a$ spanning the subalgebra $\mathcal{L}_o$

$$Q_a^j(\mathrm{x},\mathrm{u}^{(1)}) = \left\{\phi_a^j(\mathrm{x},\mathrm{u}) - \xi_a^{\mu}(\mathrm{x},\mathrm{u})\frac{\partial u^j}{\partial x^\mu}\right\}, \qquad a = 1,\,\ldots\,,r\,, \quad j = 1,\,\ldots\,,q\,. \tag{6.4}$$

We require that the subgroup $\mathcal{G}_o$ acts regularly and transversally on the manifold $\mathcal{M} = E \times U$. It means that for each point $(\mathrm{x},\mathrm{u}) \in \mathcal{M}$ the relation

$$\mathrm{rank}\big\{\xi_a^{\mu}(\mathrm{x},\mathrm{u})\big\} = \mathrm{rank}\big\{\xi_a^{\mu}(\mathrm{x},\mathrm{u}), \phi_a^j(\mathrm{x},\mathrm{u})\big\} \tag{6.5}$$

holds. When this condition, which we call the "strong transversality" condition, is not satisfied for a given subalgebra, in principle the classical SRM cannot be applied, i.e., the rank of the Jacobian matrix (3.7) is not maximal. It still possible, however, that there exists a certain domain $\mathcal{M}_o \subset \mathcal{M}$ for which this condition is fulfilled. To distinguish such cases, Grundland *et al.* [24] have introduced the notion of "weak transversality" and have shown that subalgebras with this property can still be used to construct invariant and partially invariant solutions using a specific algorithm developed for this purpose. In order to do it, we determine the conditions on $\mathrm{u} = f(\mathrm{x})$ under which rank requirement (6.5) is satisfied and solve these conditions to obtain the general form of these functions. Next we substitute the obtained expressions into the matrix of characteristics (6.4) and require that the condition $\mathrm{rank}\, Q = 0$ is satisfied. This provides further constraints on the functions $f(\mathrm{x})$. Finally, we solve the overdetermined system obtained from the system under investigation subjected to the constraints coming from requirements of weak transversality. If solutions of this overdetermined system exist then invariant solutions can be constructed explicitly.

References

[1] Alfvén H and Fälthammar C -G 1963 *Cosmic Electrodynamics* 2nd edn (Clarendon Press: Oxford)

[2] Landau L D, Lifshitz E M and Pitaevskii L P 1984 *Electrodynamics of Continuous Media* (Pergamon Press: Oxford)

[3] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (Academic Press: New York)

[4] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations, Applied Mathematical Science* (Springer Verlag: New York)

[5] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (Springer Verlag: New York)

[6] Ibragimov N H 1994/1995/1996 *Handbook on Lie Group Methods of Differential Equations* (3 volumes) (Boca Raton: CRC Press)

[7] Clarkson P A and Winternitz P 1999 Symmetry reduction and exact solutions of nonlinear partial differential equations *The Painlevé Property One Century Later* (Springer-Verlag: New York)

[8] Fuchs J C 1991 Symmetry groups and similarity solutions of MHD equations J. Math. Phys. **32** 1703

[9] Grundland A M and Lalague L 1995 Lie subgroups of symmetry groups of fluids dynamics and magnetohydrodynamics equations Can. J. Phys. **73** 463

[10] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1977 Continuous subgroups of the fundamental groups of physics J. Math. Phys. **18** 2259

[11] Picard P 2004 Sur les Solutions Invariantes et Conditionnellement Invariantes des Équations de la Magnétohydrodynamique (Ph. D. thesis) (Université de Montréal: Montréal).

[12] Grundland A M and Picard P 2004 On conditionally invariant solutions of magnetohydrodynamic equations. Multiples waves J. Nonlinear Math. Phys. **11** 47

[13] Batchelor G K 1967 *An Introduction to Fluid Dynamics* (Cambridge University Press: Cambridge)

[14] Ince E L 1956 *Ordinary Differential Equations* (Dover: New York)

[15] Ablowitz M J, Ramani A and Segur H 1980 A connection between nonlinear evolution equations and ordinary differential equations of P-type I J. Math. Phys. **21** 715

[16] Baldwin D, Hereman W and Sayers J 2004 Symbolic algorithms for the Painlevé test, special solutions, and recursion operators for nonlinear PDEs *CRM Proceedings for the Workshop on Group Theory and Numerical Methods* (AMS: Montréal)

[17] Moffatt H K 1989 *Turbulence and Nonlinear Dynamics in MHD Flows* (North-Holland: Amsterdam)

[18] Somov B V 1994 *Fundamentals of Cosmic Electrodynamics* (Kluwer Academic Publishers: Dordrecht)

[19] Swanson D G 1989 *Plasma Dynamics* (Academic Press: New York)

[20] Grundland A M and Lalague L 1994 Lie subgroups of symmetry groups of the equations describing a nonstationary and isentropic flow: Invariant and partially invariant solutions Can. J. Phys. **72** 362

[21] Gagnon L 1989 Continuous subgroups of the Galilei and Galilei-similitude groups Can. J. Phys. **67** 1

[22] Popovych V 1997 On classes of Lie solutions of MHD equations expressed via the general solution of the heat equation J. Nonlinear Math. Phys. **4** 149

[23] Grundland A M and Lalague L 1996 Invariant and partially invariant solutions of the equations describing a non-stationary and isentropic flow for an ideal and compressible fluid in $(3+1)$ dimensions J. Phys. A **29**, 1723

[24] Grundland A M, Tempesta P and Winternitz P 2003 Weak transversality and partially invariant solutions J. Math. Phys. **44**, 2704

[25] Anderson I, Fels M and Tore C 2000 Group invariant solutions without transversality Commun. Math. Phys. **212**, 653